

Analytical Solutions for Manned Lunar Landing Trajectories

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Various papers on terminal guidance^{(1)(2)(3)(4)*} are available in the literature. However, combined mid-course and terminal guidance without approximation is still a problem at the present time. In this article a continuous thrust program for a single engine is employed for the entire descent phase of a vehicle from an orbit. The weight of a retro-rocket is generally proportional to its rated thrust. The optimization problem of the combined mass of fuel and engine is approached approximately.

The guidance and control system can be carried by the space vehicle to eliminate the inherent delay of an earth-bound radio command system (2.56 seconds round trip). All necessary special computers will be sufficiently simple and small for space travel because all computational solutions are in closed algebraic form, thus enabling an astronaut to operate the vehicle and determine his own course if manual operation is preferred.

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(1) Analytical Solutions

The equations of motion⁽⁵⁾ of a space vehicle in Fig. 1 are written with the square of the specific angular momentum as the independent variable⁽⁶⁾. These equations describe the dynamical behavior of the rocket-powered vehicle in an inverse square force field, i.e.,

$$\frac{d\theta}{dk} = \frac{u^3}{2a_\theta} , \quad (1)$$

$$\text{and } \frac{d^2u}{dk^2} + \left[\frac{1}{2k} + \frac{\frac{d}{dk}\left(\frac{2a_\theta}{u^3}\right)}{\left(\frac{2a_\theta}{u^3}\right)} \right] \frac{du}{dk} + \left(\frac{u^3}{2a_\theta}\right)^2 \left[u - \frac{g_0}{ku_0^2} + \frac{a_r}{ku^2} \right] = 0 , \quad (2)$$

where $u = 1/r$ = inverse of the radial distance measured from the center of the moon (or planet) to the space vehicle,

θ = angular displacement of the vector r with respect to the local vertical of the landing point,

k = square of the specific angular momentum = $(r^2\dot{\theta})^2$,

a_r = radial specific force,

a_θ = transverse specific force,

g_0 = gravitational acceleration at a reference altitude above the moon (or planet),

$u_0 = 1/r_0$ = inverse of the radius at a reference altitude above the moon (or planet).

The solutions to these equations can be obtained provided that the specific forces are given. The transverse and radial specific forces must be chosen so that the soft landing requirements are satisfied. It is purposed that

$$\left(\frac{2a_\theta}{u^3}\right) = \frac{1}{\beta} \left(\frac{k}{k_b}\right)^n, \quad 0 \leq n < 1, \quad (3)$$

$$\left(\frac{a_r}{ku^2}\right) = \frac{r_0}{ku_0^2} - u + \lambda^2 \frac{k_b^2 q}{k^2 q} (u - u_0), \quad q \leq \frac{1}{2}, \quad (4)$$

where k_b is the initial value of k .

The constants parameters λ and β are to be determined. The justification for the limits on the parameters on n and q is that, as k approaches zero, the value of a_θ and a_r must be finite. This results in $0 \leq n$ and $q \leq \frac{1}{2}$. Under the initial conditions ($\theta = \theta_b$ at $k = k_b$) and the final conditions ($\theta = 0$ at $k = 0$), the following expressions are obtained by solving equations (1) and (3)

$$\left(\frac{\theta}{\theta_b}\right)^{\frac{1}{1-n}} = \left(\frac{k}{k_b}\right), \quad (5)$$

and

$$\beta = \frac{(1-n)\theta_b}{k_b}. \quad (6)$$

The value of β approaches zero as n approaches unity. Hence from equation (3), a_θ approaches infinity. Thus the quantity n is less than unity as given in equation (3).

After substituting equations (1), (3) and (4) into equation (2), the following linear differential equation results

$$\frac{d^2U}{dk^2} + \left[\frac{1}{2k} + \frac{n}{k}\right] \frac{dU}{dk} + \beta^2 \lambda^2 \frac{k_b^{2(n+q)}}{k^{2(n+q)}} U = 0, \quad (7)$$

where $U = u_0 - u$.

The solution must satisfy the following initial and final conditions, i.e.,

$$\text{for } \theta = \theta_b (k=k_b), \quad U = U_b \quad (r = r_b) ;$$

$$\text{and for } \theta = 0 (k = 0), \quad U = 0 \quad (r = r_0) ,$$

as shown in Fig. 1.

Thus the solution of equation (7) is a Bessel function of the first kind⁽⁷⁾⁽⁸⁾⁽⁹⁾.

$$\frac{U}{U_b} = \frac{\left(\frac{k}{k_b}\right)^\mu J_{\mu/\nu} \left[\frac{\beta \lambda k_b}{\nu} \left(\frac{k}{k_b}\right)^\nu \right]}{J_{\mu/\nu} \left[\frac{\beta \lambda k_b}{\nu} \right]}, \quad (8)$$

where $\mu = \frac{1-2n}{4}$ and $\nu = 1 - n - q$. As an example, a simple form of equation (8) is given with $n = 0$, $q = \frac{1}{2}$, thus

$$\frac{U}{U_b} = \frac{\sin 2\lambda \theta_b \left(\frac{\theta}{\theta_b}\right)^{\frac{1}{2}}}{\sin 2\lambda \theta_b} \quad (9)$$

where $\mu = \frac{1}{4}$ and $\mu/\nu = \frac{1}{2}$ in equation (8).

The above is a solution of the equation

$$\frac{d^2U}{dk^2} + \frac{1}{2k} \frac{dU}{dk} + \beta^2 \lambda^2 \frac{k_b}{k} U = 0 \quad (10)$$

(2) Trajectory Determination

By substituting equation (3) into equation (1) and integrating one obtains (with $n = 0$)

$$\frac{\theta}{\theta_b} = \frac{k}{k_b} , \quad (11)$$

where

$$k = u^{-4} \left(\frac{d\theta}{dt} \right)^2. \quad (\text{by definition}) \quad (12)$$

By combining equations (11) and (12) the angular velocity is

$$\frac{d\theta}{dt} = \left(\frac{d\theta}{dt} \right)_b \frac{u^2}{u_b^2} \frac{\theta^{\frac{1}{2}}}{\theta_b^{1/2}} . \quad (13)$$

Since $k = 0$ at $\theta = 0$ the transverse velocity at the reference point is therefore zero, i.e.,

$$\left(\frac{d\theta}{dt} \right)_{\theta=0} = 0 \quad (14)$$

which is a required condition at the terminal for vertical landing.

The value of λ is determined by the initial condition that the trajectory is tangent to the original orbit.

If the initial trajectory is tangent to a circular orbit, then

$$\left. \frac{dU}{d\theta} \right|_{\theta=\theta_b} = 0. \quad (15)$$

Differentiating equation (9) and substituting into equation (15), shows that

$$\lambda = \frac{\pi}{4\theta_b} . \quad (16)$$

For a circular orbit, k_b becomes⁽¹⁾

$$k_b = \frac{g_o}{u_o^2 u_b} . \quad (17)$$

The radial velocity can be obtained from equation (9). For a vehicle landed from a circular orbit, the velocity becomes

$$\left. \frac{dr}{dt} \right|_{\theta=0} = u^{-2} \left. \frac{dY}{d\theta} \frac{d\theta}{dt} \right|_{\theta=0} = - \frac{U_b \pi}{4\theta_b} \left(\frac{g_o}{u_o^2 u_b} \right)^{\frac{1}{2}} . \quad (18)$$

This results in a small radial velocity, which can be easily reduced to zero by introducing an additional landing phase to be discussed later. For the case of an orbital altitude of 10.909 miles⁽²⁾ and a range angle $\theta_b = 10^\circ$, the radial velocity from equation (18) is 4.5% of the orbital velocity. This is shown in Fig. 2 among other altitudes and range angles. For an orbital velocity of 3733 miles per hour this radial velocity at the reference point is only 169.7 mi./hr.

If the initial trajectory is tangent to a circular orbit, the resultant specific force may be approximated to give (see Appendix A for details)

$$\frac{a}{g_o} = \sqrt{\left(\frac{a_r}{g_o}\right)^2 + \left(\frac{a_\theta}{g_o}\right)^2} = \sqrt{\left(1 - \frac{\theta}{\theta_b}\right)^2 + \left(\frac{1}{2\theta_b}\right)^2} , \quad (19)$$

which is plotted vs. θ/θ_b in Fig. 3, from which one may conclude that the specific thrust a/g_o is nearly constant for low range angles such as $\theta_b = 5^\circ$ or 10° .

(3) Vertical Descent

Except during terminal hovering of the bug it is proposed that a constant thrust be employed for the vertical descent. It is found that, at the end of the curvilinear landing phase, the angle between the resultant thrust and the local horizontal is

$$\gamma_0 = \tan^{-1} \frac{a_r}{a_\theta} \bigg|_{\theta=0} = \tan^{-1}(2\theta_b), \quad (20)$$

which is also plotted in Fig. 2.

If the angle of the thrust vector changes instantly from $\tan^{-1}(2\theta_b)$ to $\pi/2$, and the magnitude F_0 is maintained constant thereafter, the equation of motion for the vertical descent at low altitude is

$$\frac{dv}{dt} = \frac{F_0}{m} - g_0, \quad (21)$$

where F_0 = constant magnitude of rocket thrust,

v = vertical velocity,

m = mass of the vehicle at time t ,

and g_0 = gravitational acceleration of the moon.

The rocket thrust for constant exhaust velocity c is

$$F_0 = -c \dot{m}. \quad (22)$$

Substituting equation (22) into equation (21) and integrating, yields

$$v(t) - v(0) = -c \ln(1 - \frac{a_0}{c} t) - g_0 t, \quad (23)$$

where
$$a_o = \frac{F_o}{m_o} = a \Big|_{\theta=0} = g_o \sqrt{1 + \left(\frac{1}{2\theta_b}\right)^2},$$

$$v(0) = - \sqrt{\frac{r_o}{u_o^2 u_b}} \frac{U_b \pi}{4\theta_b} = \frac{dr}{dt} \Big|_{\theta=0},$$

and m_o = initial mass at vertical descent.

The velocity must be zero at touchdown for a soft landing so that

$$v(t_f) = 0 \quad \text{at} \quad t = t_f,$$

where t_f , the time at touchdown, is obtained from equation (23).

The total distance traveled during vertical landing is shown as follows

$$s(t_f) - s_o = \frac{cv(0)}{a_o} + \frac{r_o t_f^2}{2} + c(1 - \frac{g_o}{a_o})t_f, \quad (24)$$

where $s(t_f) = 0$,

and s_o = initial altitude required for vertical descent.

The value s_o for different landing angles θ_b is plotted in Fig. 4. It shows a small distance of approximately half a mile for an orbiting altitude of 10.909 miles⁽²⁾ and a range angle $\theta_b = 10^\circ$.

From equation (22), the mass ratio is obtained, i.e.,

$$\frac{m_f}{m_o} = 1 - \frac{a_o}{c} t_f, \quad (25)$$

where m_f is the final mass.

The mass ratio vs. landing angle θ_b is shown in Fig. 5. For an orbiting altitude of 10.909 miles⁽²⁾ above the reference point, it is noted that the ratio of final mass to reference mass is about 0.96, indicating that the fuel consumption during the vertical descent is relatively very small. The optimization analysis will neglect this small change of mass.

(4) Computation of Mass Variations

For the curvilinear landing phase from orbit the rocket thrust is expressed as⁽¹⁰⁾

$$-\dot{m}c = ma, \quad (26)$$

where \dot{m} = mass flow rate, and c = constant exhaust velocity, thus

$$\frac{dm}{m} = \frac{a}{c} \frac{d\theta}{(-\dot{\theta})} = \frac{a}{c} \frac{d\theta}{\sqrt{ku^2}}, \quad (27)$$

where the quantity a is given in equation (19).

The mass ratio at low altitude may be obtained as

$$\frac{m}{m_b} = e^{-I_0}, \quad (28)$$

where

$$I_0 = \frac{g_0 \frac{u_b}{u_0}}{c \sqrt{g_0 u_b}} \int_{\theta}^{\theta_b} \left(\frac{\theta}{\theta_b} \right)^{-\frac{1}{2}} \sqrt{\left(1 - \frac{\theta}{\theta_b} \right)^2 + \left(\frac{1}{2\theta_b} \right)^2} d\theta, \quad (29)$$

m = mass at angle θ and m_b = initial mass.

For the case of lunar landing the gravitational acceleration near the surface of moon where $u_0^{-1} = 1080$ miles is

approximately $g_0 = 5.31 \text{ ft/sec}^2$. The exhaust velocity c is the product of $g = 32.2 \text{ ft/sec}^2$ and the specific impulse of the engine. If we take the specific impulse to be 311 seconds, then the corresponding value of c is 10^4 ft/sec .

The lunar bug in orbit at altitudes of 5.427, 10.909⁽¹⁴⁾ or 22.04 miles above the moon, corresponds to values of $u_b = 1085.427, 1090.909$ or 1102.04 miles, respectively. Therefore the ratio u_b/u_0 is approximately unity and the mass ratio m/m_b is insensitive to the variation of the low altitude for lunar landing in accordance with equation (28) and (29). This mass ratio m/m_b is plotted vs. θ/θ_b for various range angles θ_b in Fig. 6, where the mass variation in pounds of weight is also given for a lunar bug of 12 tons⁽¹¹⁾ while in orbit. Figure 7 shows the mass ratio m/m_0 vs. θ/θ_b where m_0 is the reference mass when the bug has no transverse velocity and descends vertically thereafter. Table 1 lists the ratios of the initial to reference mass (m_b/m_0) for various range angles θ_b and different altitudes. Here again the mass ratio is insensitive to altitude. As the quantity m_b/m_0 decreases with decreasing range angle θ_b , an asymptotic value of $m_b/m_0 = .729$ is reached as θ_b approaches zero.

TABLE 1

Values of $\frac{m_b}{m_o}$ for $g_o = 5.31 \text{ ft/sec}^2$, and $c = 10^4 \text{ ft/sec}$

θ_b	$r_b - r_o =$ 5.427 miles	$r_b - r_o =$ 10.909 miles	$r_b - r_o =$ 22.04 miles
0°	1.7312	1.7289	1.7239
5°	1.7389	1.7356	1.7315
10°	1.7623	1.7600	1.7547
15°	1.8019	1.7995	1.7939
30°	1.9834	1.9802	1.9731
45°	2.2649	2.2606	2.2508
60°	2.6378	2.6318	2.6182

TABLE 2

Values of $\frac{F_b}{m_o g_o}$ for $g_o = 5.31 \text{ ft/sec}^2$, and $c = 10^4 \text{ ft/sec}$

θ_b	$r_b - r_o =$ 5.427 miles	$r_b - r_o =$ 10.909 miles	$r_b - r_o =$ 22.04 miles
0°	"	"	"
5°	9.9633	9.9501	9.9211
10°	5.0487	5.0419	5.0269
15°	3.4594	3.4369	3.4263
30°	1.8940	1.8910	1.8841

(5) Computation of Thrust Range

The thrust developed from a rocket engine should be kept as flat as possible for the following two reasons. Firstly, the limitation of variable thrust on engine design⁽¹²⁾ will be discussed later. Secondly, the fuel consumption is usually at its minimum for a bang-bang system with constant thrust, which is the maximum effort of an engine. Since the thrust of a single engine for lunar landing has to be throttled down over a wide range for hovering near the surface of the moon, the requirement that the thrust is kept strictly constant may not be necessary, for the system to give approximately minimum fuel for the mission. Figure 8 is a plot of the thrust to reference mass ratio $F/m_0 g_0$ vs. θ/θ_b for various range angles θ_b .

Table 2 lists the values of $F_b/m_0 g_0$ which are insensitive to the change of altitude orbit and usually give the highest thrust in Fig. 8 at low range angles θ_b . Hovering above the moon's surface usually employs the same engine and requires a thrust approximately equal to the value of $m_0 g_0$, therefore the value of $F_b/m_0 g_0$ in Fig. 9 also indicates the range of variable thrust. For example, at a range angle of $\theta_b = 5^\circ$, the required thrust throttling range is about 10:1, which seems to be the upper limit of practical operation at this time⁽¹²⁾. If the manufacturer's specifications for a particular engine indicate a certain maximum throttling range, any value of θ_b below the corresponding value of $F_b/m_0 g_0$ will not be considered. For example, if $F_b/m_0 g_0 = 10$, then from

Fig. 9 or Table 2, the range angle θ_b should be larger than 5° .

(6) Minimization of Combined Mass of Fuel and Engine

From Table 2 the required thrust for zero range angle ($\theta_b = 0$) is infinite which is not physically realizable because it requires an infinitely large engine. The mass of the engine is approximately proportional to its rated maximum thrust. One can write

$$m_e = n \frac{T_{\max}}{c_0} \quad \text{or} \quad n = \frac{m_e c_0}{T_{\max}}, \quad (30)$$

where m_e = mass of engine

and T_{\max} = rated maximum thrust.

Efficient large engines such as that for Saturn C-1 produce $1.5(10)^6$ lb of thrust with an empty vehicle weight of 65 tons⁽¹³⁾. It is estimated that the propulsion is 4% of the total weight⁽¹⁴⁾ and the payload and structure are 8% of the total weight. Thus a 22 ton engine on earth delivering 750 tons of thrust will give

$$n = \frac{22}{750} \left(\frac{5.31}{32.2} \right) = .00484.$$

Saturn C-5 with a thrust of $7.5(10)^6$ lb has an empty vehicle weight of 170 tons. This is equivalent to

$$n = \frac{57}{3750} \left(\frac{5.31}{32.2} \right) = .00251.$$

It is estimated that a small engine with a thrust of $(10)^5$ lb may have a much larger value for n in the neighborhood of $.01 < n < .02$.

The total mass of a lunar bug consists of payload, fuel and engine. Maximizing the payload is equivalent to minimizing the combined mass of fuel and engine. The fuel mass m_u is the difference of the initial mass m_b and the reference mass m_o , i.e.,

$$m_u = m_b - m_o, \quad (31)$$

where the engine mass m_e is given in equation (30). Thus the combined mass,

$$M = m_u + m_e,$$

or

$$\frac{M}{m_o} = \left(\frac{m_b}{m_o} - 1 \right) + \eta \frac{\tau}{m_o g_o} \quad (32)$$

is the quantity to be minimized.

Figure 9 is a plot of M/m_o vs. θ_b for various values of η . The minimum value of M/m_o for $\eta = 0.01$ is at $\theta_b = 9^\circ$ while for $\eta = 0.02$, $\theta_b = 12^\circ$, provided these range values of $\frac{F_b}{m_o g_o} = 5.58$ and 4.23, respectively, are allowable as explained in the last section.

Table 3 shows the data for a typical lunar landing satisfying all the previous requirements.

One concludes from Table 3 that a 12 ton bug in orbit would have a payload of $13,100 - 688 = 12,412$ lb (earth's weight) at about the time of landing (mass consumption during hovering excluded).

TABLE 3

$$\begin{aligned} g_o &= 5.31 \text{ ft/sec}^2 & c &= 10^4 \text{ ft/sec} \\ g_e &= 32.17 \text{ ft/sec}^2 & \theta_b &= 10^\circ \\ & & n &= .01 \end{aligned}$$

	End of orbiting starts curvi- linear descent	End of Braking starts vertical descent	Terminal Point (on surface of moon)
radius	$r_b = 1090.91 \text{ miles}$	$r_o = 1080 \text{ miles}$	$r_f = 1079.47 \text{ miles}$
transverse velocity	$V_{\theta b} = 3733.02 \text{ mi/hr}$	$V_{\theta o} = 0$	$V_{\theta f} = 0$
radial velocity	$V_{rb} = 0$	$V_{ro} = 169.7 \text{ mi/hr}$	$V_{rf} = 0$
thrust	$F_b = 11,349 \text{ lb}$	$F_o = 6823 \text{ lb}$	$F_f = 6823 \text{ lb to}$ 2250 lb
thrust angle	γ_b is small	$\gamma_o = 19.42^\circ$ switch to $\gamma_o = 90^\circ$	$\gamma_f = 90^\circ$
gross weight	$m_b g_e = 24,000 \text{ lb}$	$m_o g_e = 13,640 \text{ lb}$	$m_f g_e = 13,100 \text{ lb}$
engine weight	$m_e g_e = 688 \text{ lb}$	$m_e g_e = 688 \text{ lb}$	$m_e g_e = 688 \text{ lb}$
angular error	$(\Delta \theta)_b = \Delta \theta_b$	$(\Delta \theta)_o = \Delta \theta_b$	$(\Delta \theta)_f = \Delta \theta_b$
radial error	$(\Delta u)_b = \Delta u_b$	$(\Delta u)_o = \Delta u_b$	$(\Delta u)_f = \Delta u_b$
transverse velocity error	$(\Delta V_\theta)_b$ is arbitrarily small	$(\Delta V_\theta)_o = 0$	$(\Delta V_\theta)_f = 0$
radial velocity error	$(\Delta V_r)_b$ is arbitrarily small	$(\Delta V_r)_o = 0$	$(\Delta V_r)_f = 0$
time elapse	$t_b = 0$	$t_o = 354 \text{ sec}$	$t_f = 386 \text{ sec}$

Conclusions

- (a) The trajectory may be planned to guide the lunar bug continuously from orbit to landing. Nonlinear guidance and control are required to specify rocket propulsion. Solutions for the trajectory are in closed algebraic form, thus enabling an astronaut to compute the solutions on board if he prefers manual operation. Elaborate tracking is not required as there are no tracking stations on the moon.
- (b) Due to design limitations a single rocket-engine of variable thrust is used for the descent phase of the lunar bug. As the range angle θ_p becomes small, the required thrust increases rapidly. This in turn increases the weight of the engine. There exists an optimum angle for θ_p where the effect of increasing weight due to thrust counter-balances the decrease of fuel mass.
- (c) Variation of the maximum thrust of the rocket engine is studied for a lunar soft-landing from low altitude for approximately 5° to 60° of range angle. The requirement of thrust variation places a lower bound on the range angle for the design problem.

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Appendix A

Justification is to be found for simple approximate forms for the specific forces in order to obtain an estimate of the mass ratio.

The quantity $\frac{U_b}{u_o}$ is equal to $(\frac{r_b - r_o}{r_b})$ which is much less than unity for low altitude. For example, an altitude of 22.04 miles above the moon corresponds to $\frac{U_b}{u_o} = 0.02$. Therefore the following relation obtained from equation (9) is justified:

$$\frac{u}{u_o} = 1 - \frac{U_b}{u_o} \frac{\sin 2\lambda_b \left(\frac{\theta}{\theta_b}\right)^{\frac{1}{2}}}{\sin 2\lambda_b} \approx 1. \quad (A1)$$

Substituting equation (A1) into equations (3) and (4) for $n = 0$, one obtains

$$a_\theta \approx \frac{u_o^3}{2\beta}, \quad (A2)$$

and

$$a_r \approx g_o - k u_o^3 \quad (A3)$$

where the last term in equation (4) is very small.

Combining equations (5), (6) and (17) for $n = 0$ yields

$$k = \frac{\theta}{\theta_b} \frac{r_o}{u_o^2 u_b}, \quad (A4)$$

and

$$\beta = \theta_b \frac{u_o^2 u_b}{r_o}. \quad (A5)$$

Substituting equations (A4) and (A5) into equation (A3) and equation (A2), respectively, and considering the approximation in equation (A1), the following relations are obtained:

$$\frac{a_{\theta}}{g_0} \approx \frac{1}{2\theta_b} , \quad (A6)$$

and

$$\frac{a_r}{g_0} \approx 1 - \frac{\theta}{\theta_b} . \quad (A7)$$

The resultant specific force becomes

$$\frac{a}{g_0} = \sqrt{\left(\frac{a_r}{g_0}\right)^2 + \left(\frac{a_{\theta}}{g_0}\right)^2} = \sqrt{\left(1 - \frac{\theta}{\theta_b}\right)^2 + \left(\frac{1}{2\theta_b}\right)^2} . \quad (A8)$$

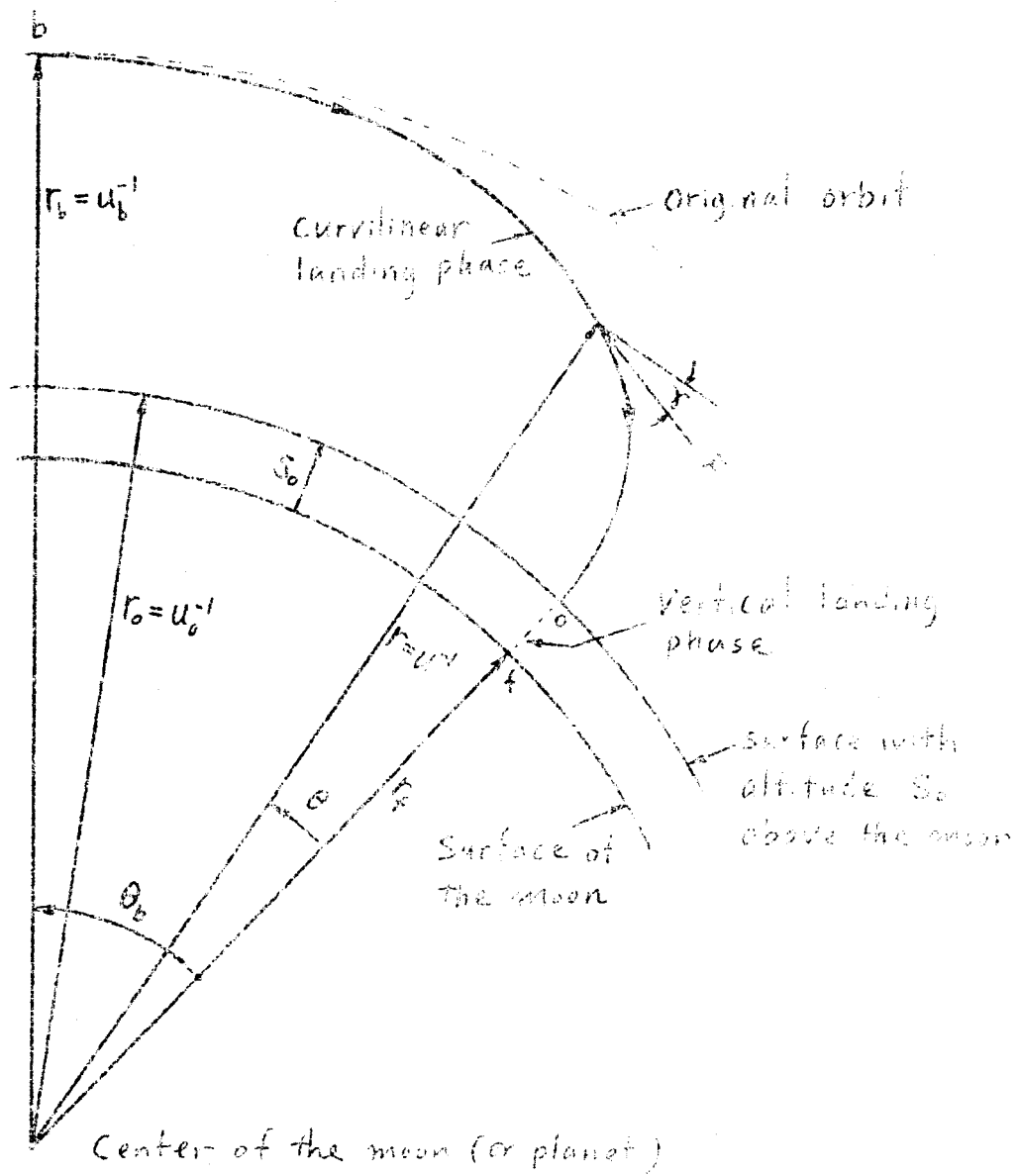


Fig 1 Descent Trajectory

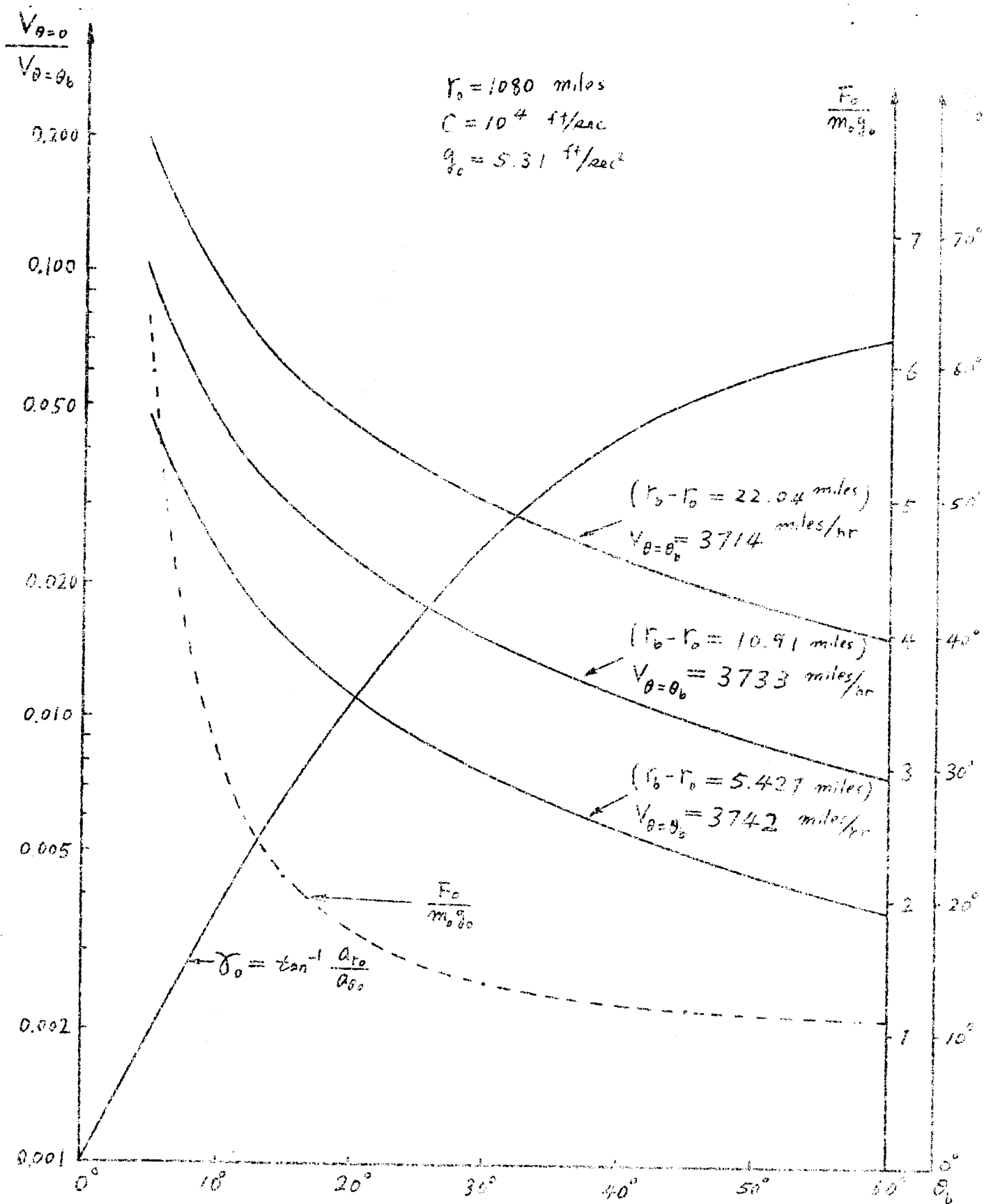


Fig 2 Velocity Ratio, Thrust to Weight Ratio
 and Thrust Angle at Reference point o
 vs θ_b

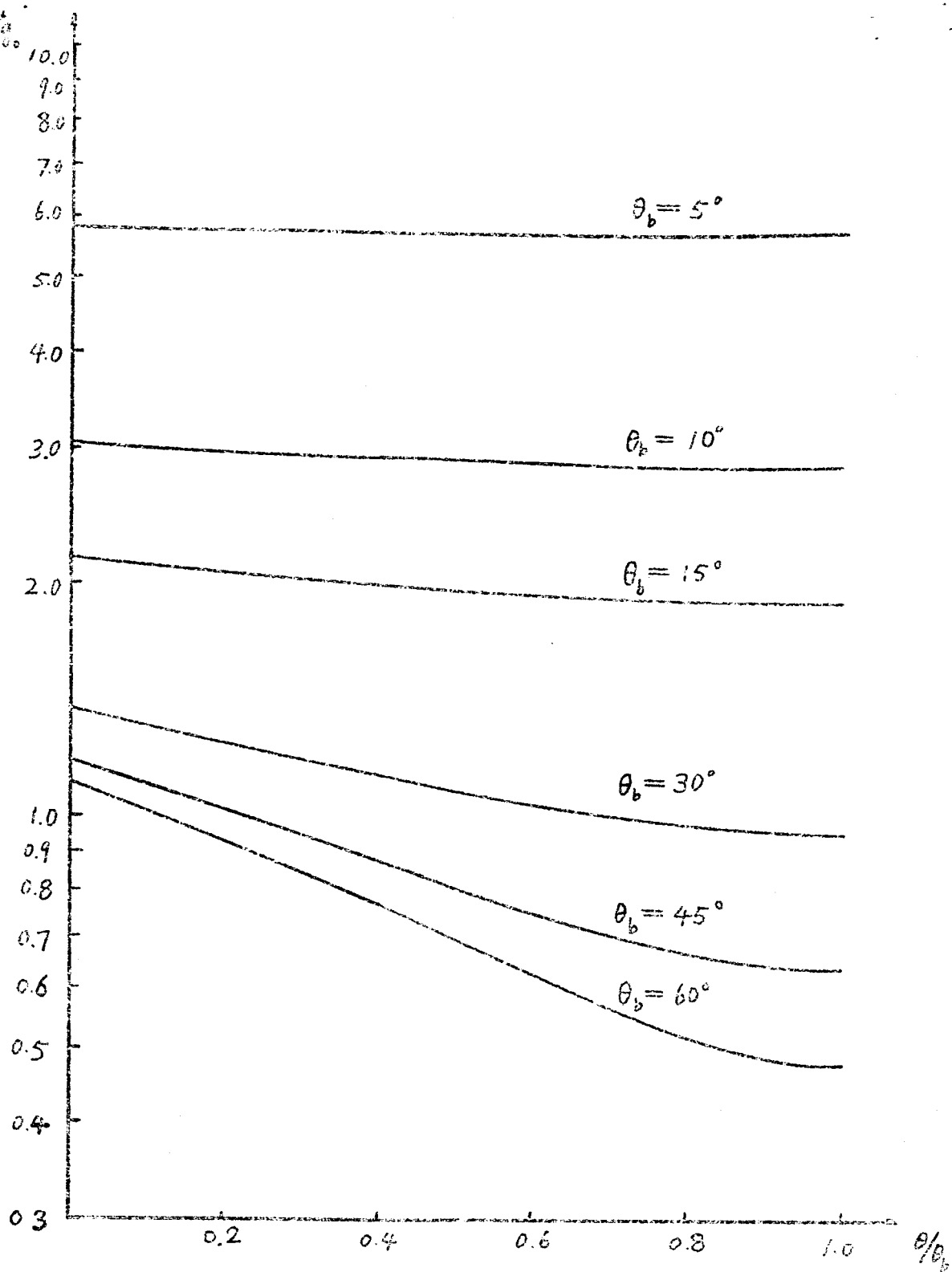


Fig 3 Specific Force vs θ/θ_b

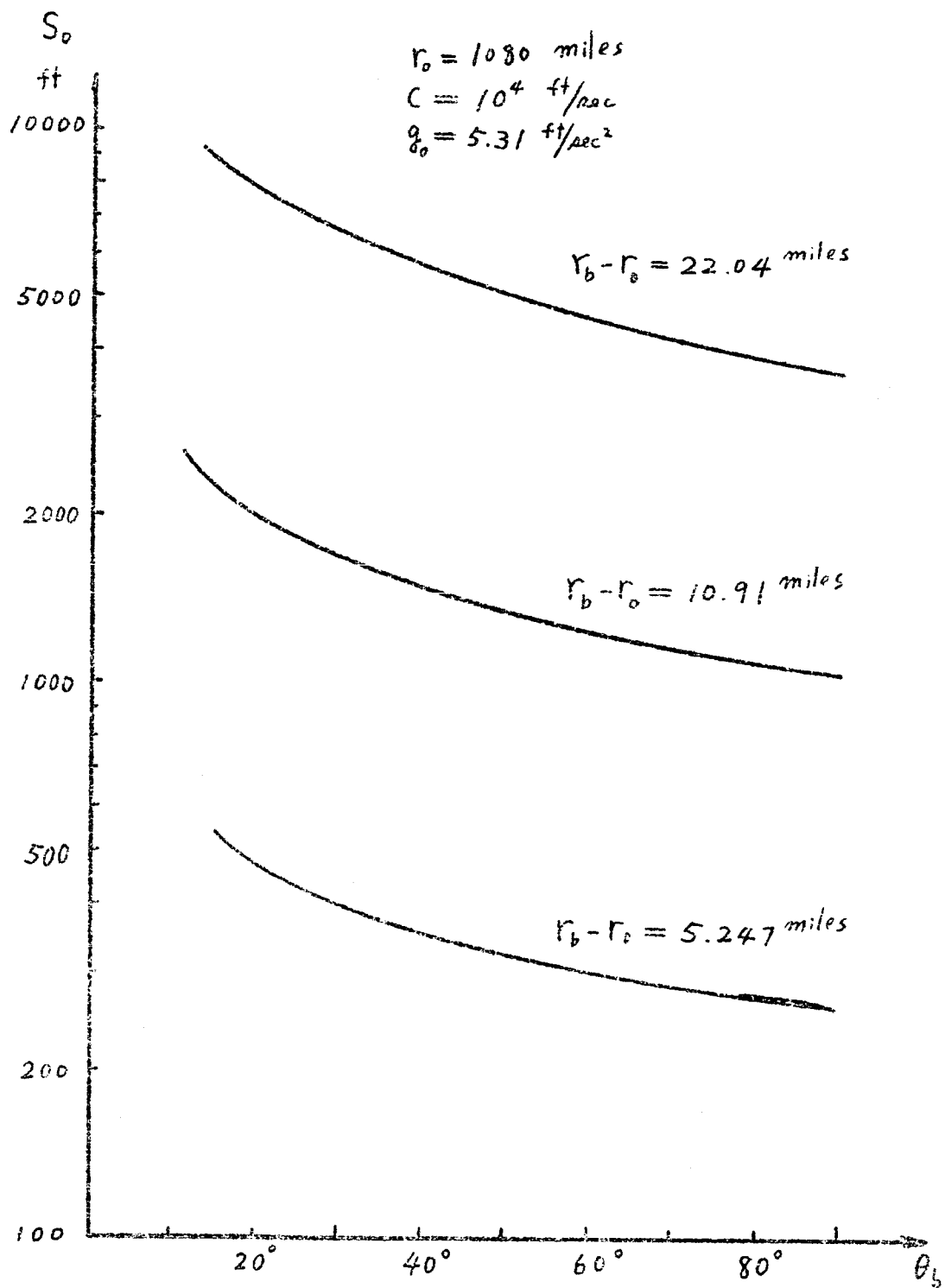


Fig 4 Initial Altitude of Vertical Descent
vs θ_b

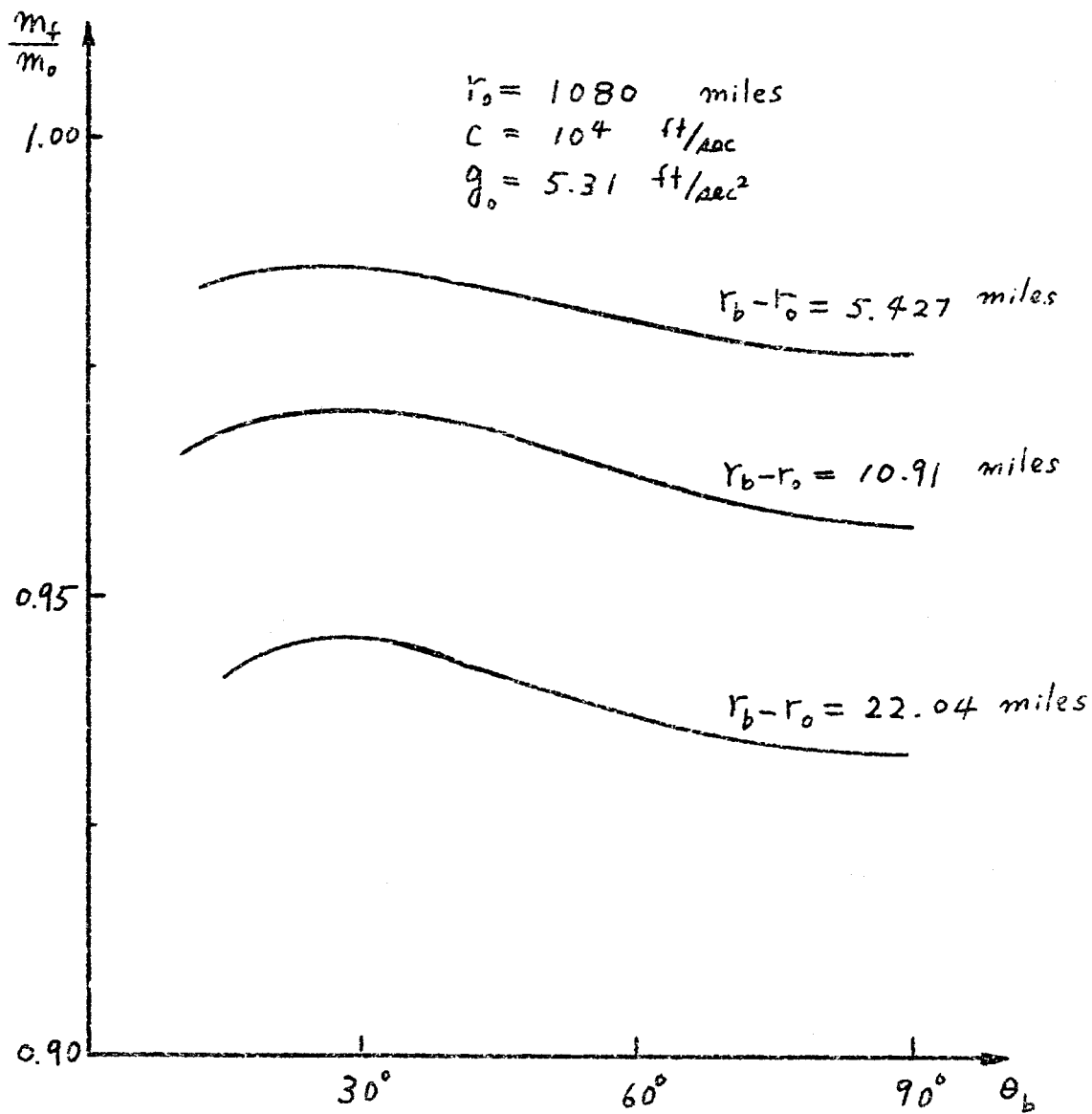


Fig 5 Mass Ratio in Vertical Descent vs θ_b

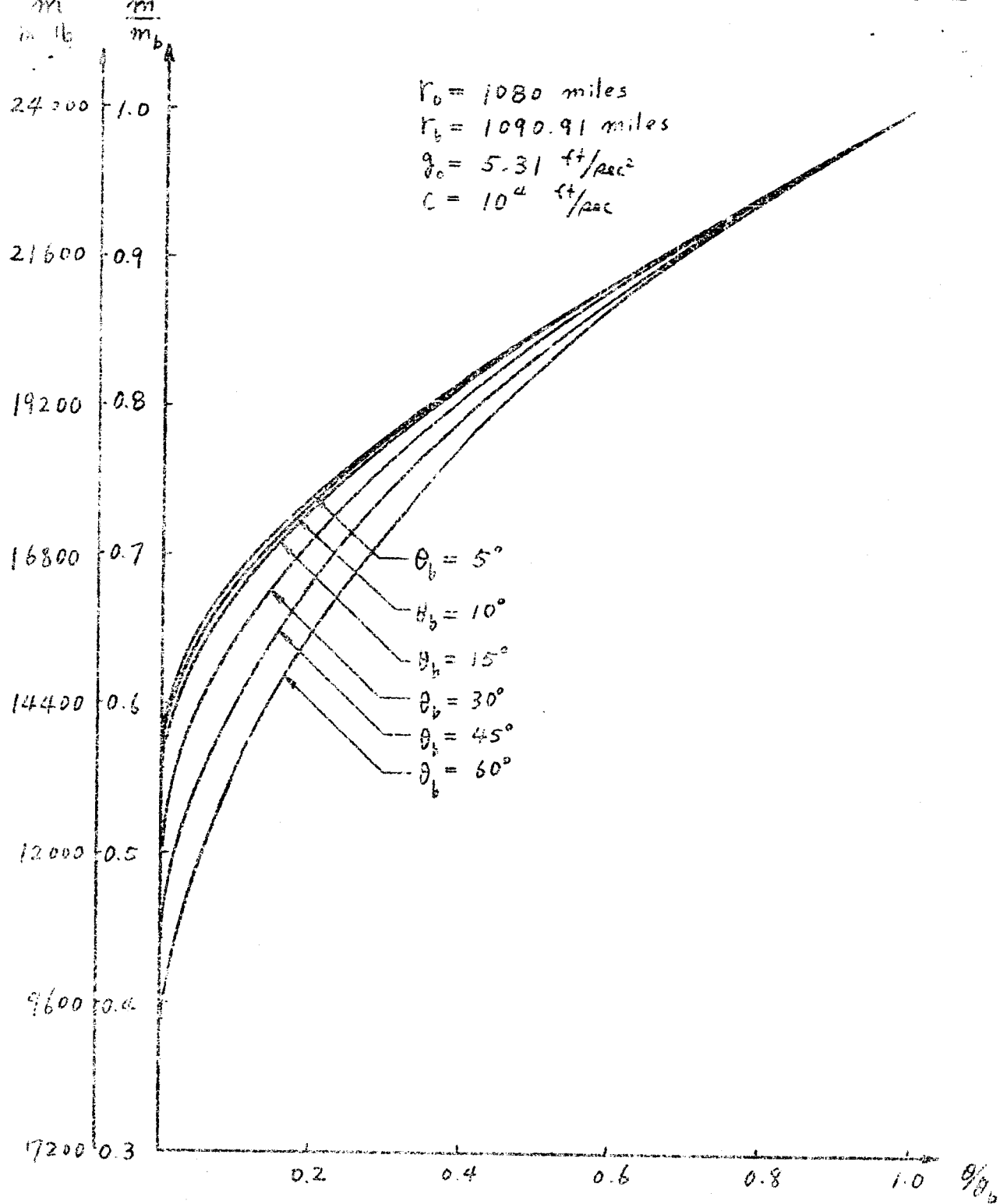


Fig 6 Mass Ratio vs θ/θ_b
 ($m_b = \text{initial mass}$)

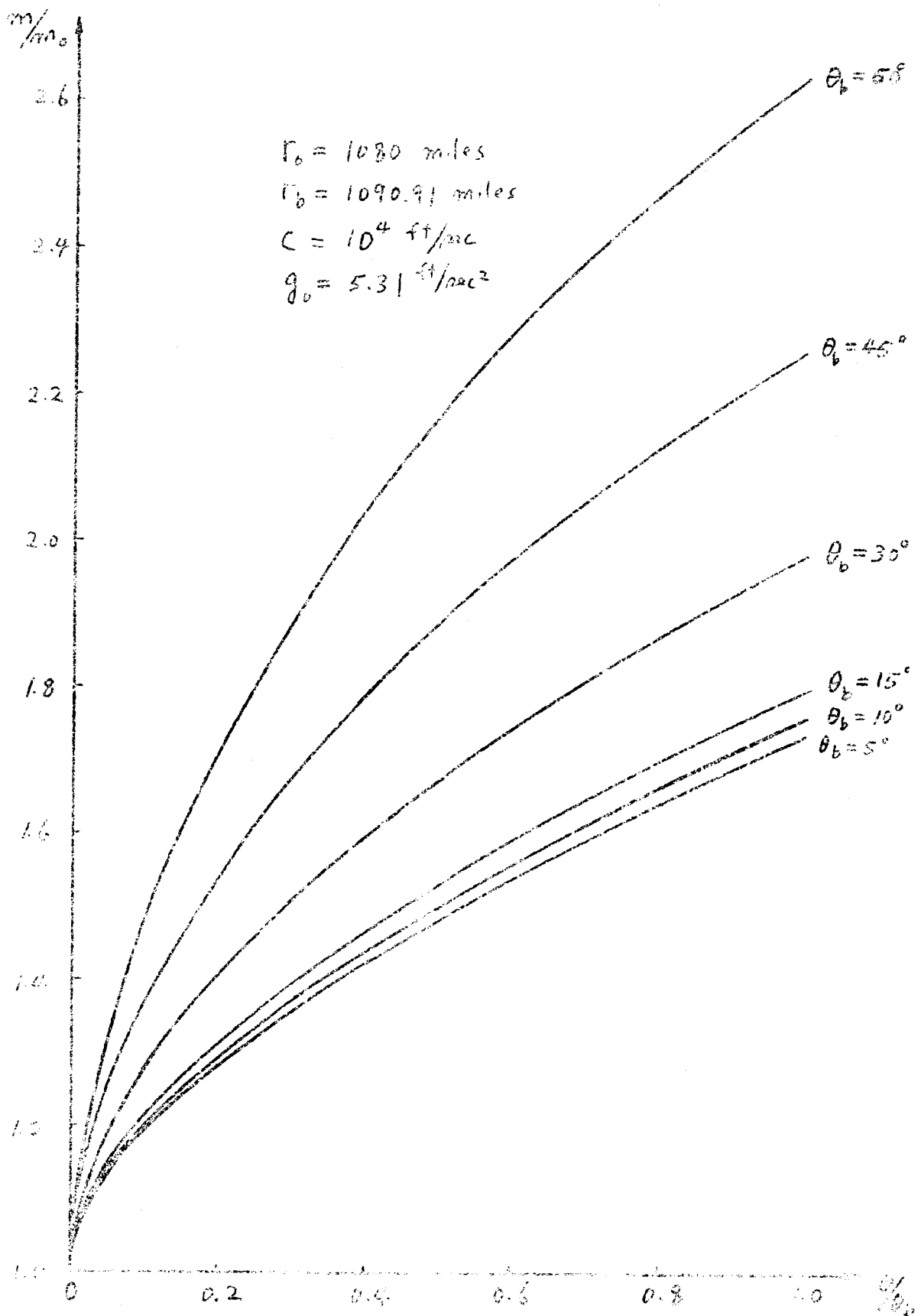


Fig 7 Mass Ratio vs θ/θ_0
 ($m_0 = \text{reference mass}$)

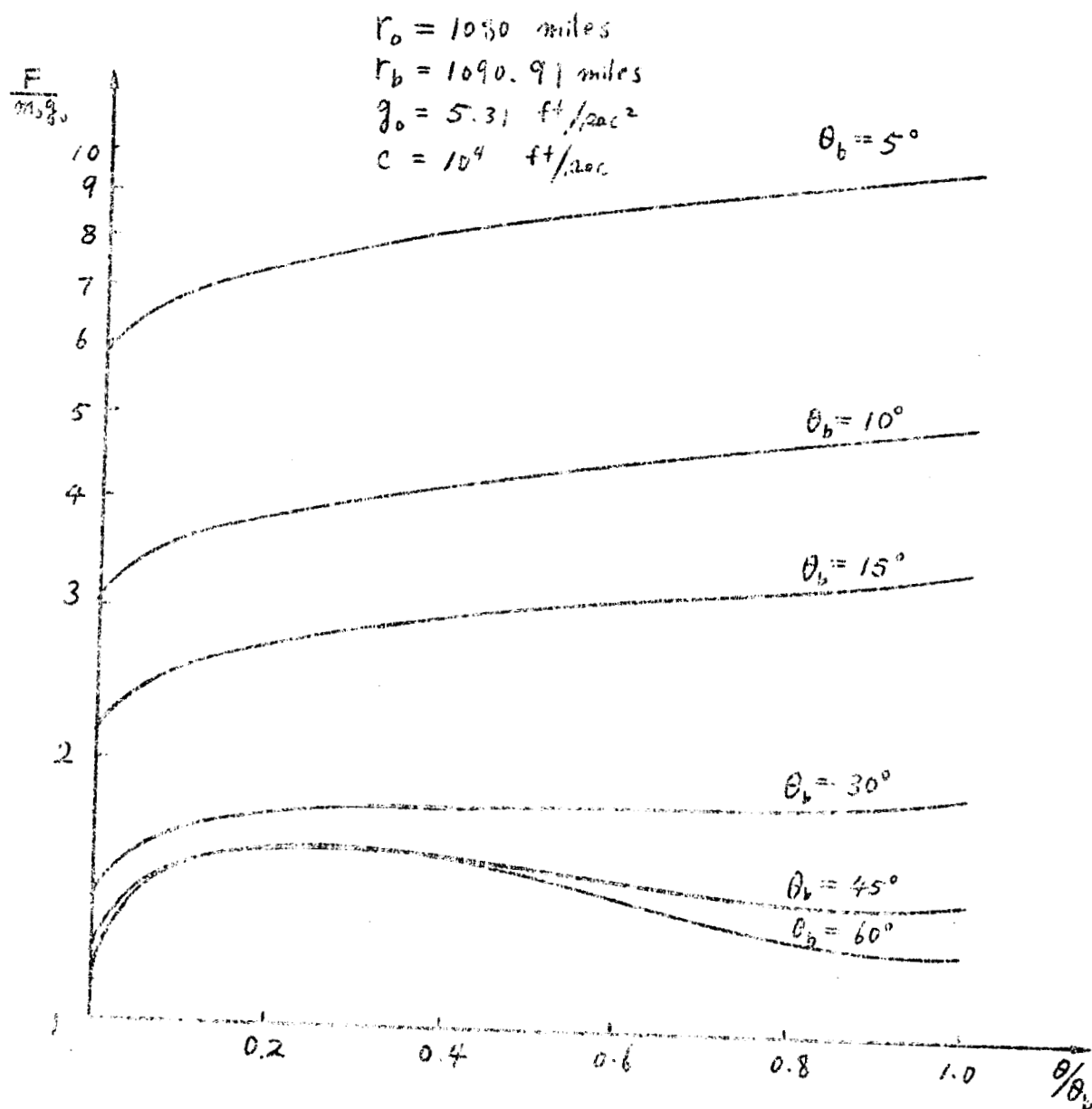


Fig 8 Thrust to Reference Weight Ratio vs θ/θ_b

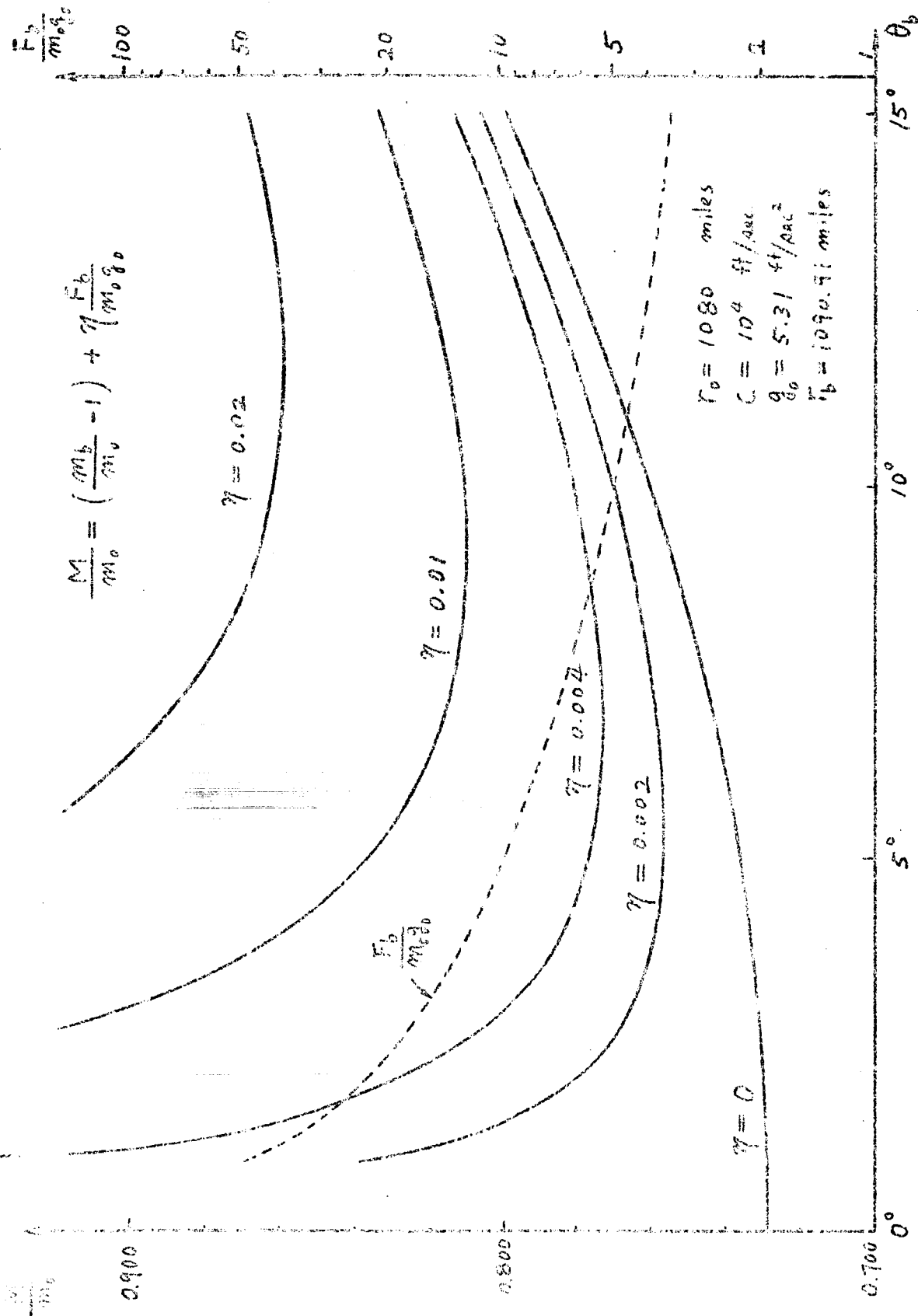


Fig 9 Combined Mass of Fuel and Engine vs θ_b